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Washington Square College of Arts and Science
Mathematics Research Group

Research Report No. EM-28

THE DIPOLE FIELD
in an
INHOMOGENEOUS ATMOSPHERE

by

Bernard Friedman

Written by:

Bernard Friedman

Morris Kline
Morris Kline
Project Director

Title page
56 numbered pages

March, 1951

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Abstract

The mathematical theory of propagation for stratified media above a spherical earth with an arbitrary scalar radial variation of the index of refraction is presented so as to be applicable to both tropospheric and ionospheric problems. The method of transformation of contours used by Watson, Rydbeck and others for special variations is applied to the general case. Some mathematical questions concerning the continuous spectrum, which are usually neglected are examined in detail. The methods of Furry, Peckeris and others for the approximate theory of tropospheric propagation, which assumes a flat earth and a modified index of refraction, is shown to follow from the general theory by making suitable assumptions. The practical problem is reduced to that of finding the eigenvalues of an ordinary differential equation. The W.K.B. method is applied to obtain explicit results for a ductless atmosphere. For the troposphere a new method of obtaining an approximation in the case of a duct is obtained which appears to be suitable for quantitative work.

1. Introduction.

The problem of explaining the propagation of radio waves around the surface of the earth has long been of great interest to both physicists and mathematicians because of its practical importance and because of its considerable mathematical difficulties.

There are two fundamental approaches to the theory of such propagation. The first approach, which is that of Watson [1], Van der Pol and Bremmer [2], and Rydbeck[3], and which has been applied mostly to ionospheric problems, assumes the earth to be a perfect sphere with finite or infinite conductivity, and expresses the field due to a radiating dipole as an infinite series involving spherical Bessel functions. Since the series converges much too slowly for practical purposes, it is transformed into a contour integral which can be evaluated by the method of residues. To find the residues of this integral, highly complicated asymptotic expansions for the spherical Bessel functions must be used. These asymptotic expansions, which are derived by the method of stationary phase, involve the Hankel functions of order 1/3 or the Airy integral, both of which occur frequently in diffraction theory.

The second approach to the problem is that used by Pryce[4] and Booker [5] in England, and by Furry [6] and Pekeris [7] in the United States while studying the effect of the troposphere on radio wave propagation. In this approach, the earth is assumed to be flat but, following a suggestion of Schelling and Burrows [8], the index of refraction is modified so as to take into account the curvature of the earth. The field due to a radiating dipole is now expressed as an infinite integral involving ordinary Bessel functions. This integral, also, can be evaluated by the method of residues but it was realized that the residues are really the successive terms in an eigenfunction expansion for the field of the dipole. The propagation problem is therefore reduced to the study of an eigenvalue problem for an ordinary differential equation. In case the atmosphere above the earth is assumed to have a constant refractive index, the eigenfunction expansion is in terms of Hankel functions of order 1/3, exactly the same functions which appear in the first approach.

In this paper we start with Maxwell's equations and then present a complete exposition of the theory of propagation in an inhomogeneous atmosphere where the dielectric constant is stratified in a radial direction. The theory can be applied to both ionospheric and tropospheric problems. The methods we have used and the results we have obtained are similar to those contained in the papers by Watson [1]

Van der Pol and Bremmer [2], and in the books by Bremmer[11] and Rydbeck [3]. The novelties of our treatment and results are the following:

1) The dielectric constant is considered to be a single function of position instead of different functions in different regions of space. As a result, the formulas for the Hertz potential and the electro-magnetic field have a simpler appearance and are not as complicated and clumsy as in the usual treatment. All the necessary transformations and approximations can be carried out on these formulas and because of their simpler appearance it is much easier to acquire an insight into their meaning. Of course, when quantitative results are wanted, it is necessary to consider the dielectric constant as composed of different functions in different regions, solve the appropriate differential equation in each region and then match the solutions across the boundary to obtain the complete solution.

2) The expression of the Hertz potential as a contour integral and the transformation of the contour integral into a sum of residues are carefully discussed. There are two questions in this procedure which are usually ignored. One is the question of whether the contour integral over an infinite semi-circle may be neglected. Watson[1] has shown that in the case of a constant atmosphere these integrals vanish. However, his proof depends upon the use of asymptotic formulas for the Bessel functions. We show in the Appendix that, even for a completely arbitrary atmosphere, these integrals vanish. The second question is this: In general, the contour integral can be expressed as a sum of residues plus an integral around the imaginary axis; when can this integral be neglected? Watson [1] shows that, in the case of the constant atmosphere, this integral is negligible compared to the residue series. Later writers on this topic either drop the integral without comment, or change the boundary condition at the center of the earth[11], or express the Hertz potential in terms of incident and reflected waves[2], which result only in a residue series and not in an integral. This latter procedure can be applied to the general case but it is only in the case of the constant atmosphere that the integral over the imaginary axis does not appear. In the appendix we consider an arbitrary atmosphere and show, first, that if the earth is assumed to be a perfect conductor, the integral over the imaginary axis vanishes identically; and, second, that if the earth is not assumed to be a perfect conductor but if its dielectric constant is assumed to be large compared to that of the atmosphere, which is usually the case in practice, then the integral over the imaginary axis is small compared to the first term of the residue series.

3) The general nature of our study emphasizes the fact that the propagation problem reduces to the solution of the eigenvalue problem for an ordinary differential equation. This fact, while it is clearly recognized in the flat earth theory and while it is implicitly contained in most previous work, has heretofore not been explicitly stated for a spherical earth. Once this fact is known, it is clear that the W.K.B. method should be applied. By means of this method, the "phase integral" method of Eckersley[10] can be understood and made more exact.

4) It is shown that the flat earth theory follows from the exact spherical earth theory by making suitable approximations in the ordinary differential equation mentioned in 3). From the nature of the approximations, it is concluded that the eigenvalues will be closely approximated but that the expressions for the field on the flat earth theory will be inaccurate at large distances above the earth. These conclusions agree with those of Pekeris[7] in his analysis of the accuracy of the flat earth approximation.

5) In the case of tropospheric propagation the W.K.B. method gives explicit results. First, if the atmosphere is inhomogeneous but does not have a duct, the field can be obtained by a change of scale from that produced by a constant atmosphere. This result generalizes a similar result obtained by Eckersley[10] and Bremmer[11] for a special form of variation of the dielectric constant. Second, if the atmosphere does have a duct, the differential equation that must be considered will have two transition points and its solution will depend on confluent hypergeometric functions. We find that the differential equation can be approximated by one which has a double transition point and that then the solution can be expressed in terms of Hankel functions of order one-fourth. This approach seems to be completely new and suggests a simple, direct quantitative method for approach to the problem of propagation in a duct.

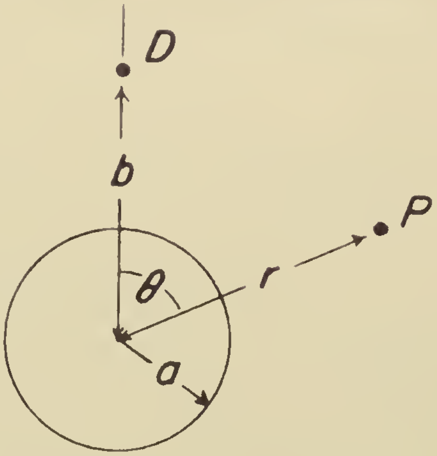
The problem of propagation in an inhomogeneous atmosphere is formulated mathematically in section 2 and it is shown that the problem requires the solution of a scalar partial differential equation. In Section 3 this equation is solved by the classical method of separation of variables and expansion in terms of Legendre polynomials. In Section 4 this solution is represented as a contour integral and evaluated by the calculus of residues. These residues are interpreted in Section 5 as an expansion of the solution in terms of radial eigenfunctions. In Section 6 we use the fact that the wave length of the radiation is very small compared to the radius of the earth to obtain simpler boundary conditions at the earth's surface and a simpler expression for the solution. We return in Section 7 to the discussion of the radial eigenfunctions and we show how, by the use of Langer's extension of the

W.K.B. method, useful approximations to the eigenfunctions can be found.

The case of a uniform atmosphere is discussed by this method in Section 8 and the well known results are obtained. Section 9 shows that the case of the non-uniform atmosphere without ducts can be treated almost as easily as the case of a uniform atmosphere. In Section 10 the "flat earth" theory and the modified index of refraction are shown to be reasonable approximations to the exact theory. Section 11 discusses the case of an atmosphere with ducts; Section 12 summarizes the work for the electric dipole. The detailed mathematical questions concerning the countour integral representation are discussed in the appendices.

2. Formulation of the Problem.

Assume the earth is a sphere, radius a, with complex dielectric constant ϵ_1. Introduce spherical polar coordinates, r, θ, ∅, where r is the distance from the center of the earth, θ is the co-latitude and ∅ the longitude. We assume that the earth is immersed in a medium whose dielectric constant, ϵ, is a function of r alone. The propagation problem depends on finding the field produced by a unit dipole which is



radiating at a constant angular frequency ω and is located at a distance, b, from the center of the earth. By a suitable rotation of the coordinate system we can always locate the dipole on the polar axis as indicated in the above diagram.

The electromagnetic field produced by the dipole should satisfy Maxwell's equations, both inside and outside the earth, and should possess at the point D a singularity corresponding to that of a dipole while on the earth's surface the tangential components of the electric field $\underline{E}$ and the magnetic field $\underline{H}$ must be continuous.

Let $\underline{E}e^{-i\omega t}$ represent the vector electric field intensity and $\underline{H}e^{-i\omega t}$ the vector magnetic field intensity produced by the dipole, then Maxwell's equations in M.K.S. units become the following:

$$\nabla \times \underline{E} = i\omega\mu \underline{H} \quad , \qquad \nabla \cdot \underline{H} = 0$$

$$(1) \qquad \nabla \times \underline{H} = -i\omega\epsilon \underline{E} \quad , \qquad \nabla \cdot \epsilon \underline{E} = 0 \quad .$$

Here, $\epsilon = \epsilon(r)$, is a scalar function which represents the variation of the dielectric constant of the medium. For $r < a$, $\epsilon(r)$ is assumed to be a constant, ϵ_1, representing the complex dielectric constant of the earth.

We shall derive the well known fact[12] that in spherical coordinates the electromagnetic field given by (1) can be obtained from a Hertz vector which has only its radial component non-zero, even when the dielectric constant is a function of r. Since $\epsilon\,\underline{E}$ is a vector whose divergence is zero, we may write

$$(2) \qquad \epsilon\,\underline{E} = i\,\omega\mu\nabla\times\epsilon\,\underline{C} \quad .$$

Substituting this expression for $\underline{E}$ in the curl $\underline{H}$ equation, we get

$$\nabla\times(\underline{H} - \omega^2\mu\,\epsilon\,\underline{C}) = 0 \; ,$$

so that the expression in parenthesis must be the gradient of some scalar, which we call ψ. We have then

$$(3) \qquad \underline{H} = \omega^2\mu\,\epsilon\,\underline{C} + \nabla\psi \; .$$

Using (2) and (3) in the remaining equations of (1), we obtain the following equations for $\underline{C}$ and ψ:

$$(4) \qquad \nabla\times\frac{1}{\epsilon}(\nabla\times\epsilon\,\underline{C}) - \omega^2\mu\,\epsilon\,\underline{C} - \nabla\psi = 0$$

$$(5) \qquad \nabla\cdot(\omega^2\mu\,\epsilon\,\underline{C}) + \nabla^2\psi = 0 \quad .$$

Note that if we take the divergence of equation (4), we obtain equation (5) so that it is sufficient to satisfy equation (4).

Let us express these equations in spherical coordinates. Since ϵ is a function of r only and since the dipole is on the polar axis, the electromagnetic field and therefore also the Hertz vector $\underline{C}$ will not depend on the longitude ϕ but only on the coordinates r, θ. We assume that the vector $\underline{C}$ has a radial component of the form rU(r,θ) where U(r,θ) is some scalar function, and that its other components are zero. Now, the θ-component of (4) implies that

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial^2}{\partial r\,\partial\theta}(rU)$$

so that we may take

$$\psi = \frac{\partial(rU)}{\partial r}$$

and then the r-component of (4) reduces exactly to the following:

(6) $$\Delta U + k^2 U = 0$$

where

(7) $$k^2 = \omega^2 \epsilon \mu$$

and Δ is the Laplacian in spherical coordinates.

The field components can be obtained from equations (2) and (3). We find that

(8) $$H_r = \left(k^2 + \frac{\partial^2}{\partial r^2}\right)(rU), \qquad H_\theta = \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta}(rU) \quad ,$$
$$E_\phi = -i\,\omega\mu\,\frac{\partial U}{\partial \theta} \quad , \qquad E_r = E_\theta = E_\phi = 0 \quad .$$

The equations in (8) represent the field of a magnetic dipole. There are similar expressions for the field due to an electric dipole. The modifications of the theory for this case are detailed in Section 12.

The problem of the propagation of radio waves around the earth now has been reduced to that of solving the scalar wave equation (6). We know that

$$k^2(r) = \text{constant} = \omega^2 \epsilon_1 \mu \quad , \; r < a$$

$$k^2(r) = \omega^2 \epsilon(r) \mu \quad , \qquad r > a$$

and that, at $r = a$, U and $\frac{\partial U}{\partial r}$ must be continuous because the tangential components of the field given by (8) must be continuous. We know also that U must be regular everywhere except at the dipole, that is, at $r = b$, $\theta = 0$. Finally, we know that the dipole is sending out radiation so that all the energy is going to infinity. This is expressed mathematically by the Sommerfeld radiation condition. Before writing it down, we remark that since the medium must eventually become free space, the value of ϵ will approach that of free space which we designate by ϵ_0. The Sommerfeld radiation condition may now be expressed as follows:

(9) $$\lim_{r\to\infty} r \left| \frac{\partial U}{\partial r} - ik_0 U \right| = 0$$

where

(10) $$k_0^2 = \omega^2 \epsilon_0 \mu \quad .$$

This condition means that, at infinity, U behaves like $e^{ik_0 r}/r$ so that we are dealing with an outgoing wave.

3. The Series Solution for U(r,θ).

The fact that U(r,θ) possesses a dipole singularity at $r = b$, $\theta = 0$ while otherwise it satisfies the wave equation (6) can be expressed very concisely by using the Dirac δ-function. We write

$$\Delta u + k^2(r)U = - \frac{\delta(r-b)\,\delta(\theta)}{2\pi\, r^2 \sin\theta} \tag{11}$$

where the denominator $2\pi\, r^2 \sin\theta$ has been introduced to normalize the solution. It is essentially the Jacobian of the transformation from rectangular to spherical coordinates.

Since U(r,θ) is a regular function of (r,θ) except at $r = b$, $\theta = 0$, we may assume

$$U = \sum_{n=0}^{\infty} \frac{2n+1}{2}\; u_n(r)\, P_n(\cos\theta) \quad . \tag{12}$$

From (12) by using the well known orthogonality properties of the Legendre polynomials we find that

$$u_n(r) = \int_0^{\pi} U(r,\theta)\, P_n(\cos\theta)\sin\theta\, d\theta \quad . \tag{13}$$

Now, multiply (11) by $P_n(\cos\theta)\sin\theta\, d\theta$ and integrate between 0 and π. The equation transforms into the following ordinary differential equation:

$$\frac{1}{r}\,\frac{d^2(ru_n)}{dr^2} + \left[k^2(r) - \frac{n(n+1)}{r^2}\right] u_n = - \; - \frac{\delta(r-b)}{2\pi r^2} \; ; \tag{14}$$

or

$$\frac{d^2(ru_n)}{dr^2} + \left[k^2(r) - \frac{n(n+1)}{r^2}\right](ru_n) = - \frac{\delta(r-b)}{2\pi r} \quad .$$

Let us investigate the meaning of (14). If we integrate it first from $r = r_0$ to $r = b - h$ and then from $r = r_0$ to $r = b + h$, h a small quantity, we find that $\frac{d(ru_n)}{dr}$ has a jump of magnitude $- \frac{1}{2\pi b}$ as r goes from $b - h$ to $b + h$,

while ru_n remains continuous. Note that the point $r = a$ causes no trouble because, there, just as for every value of $r \neq b$, the solution ru_n and its derivative must remain continuous. (These conditions are, of course, the same as those which have already been imposed on $U(r,\theta)$.) The other conditions on U, that it be regular at $r = 0$ and that it satisfy the Sommerfeld radiation condition (9), imply that u_n is regular at $r = 0$ and that

$$(15) \qquad \lim_{r\to\infty} \left| \frac{d(ru_n)}{dr} - i\, k_o\, ru_n \right| = 0 \quad .$$

In order to solve (14) we must know the solutions of the homogeneous equation:

$$(16) \qquad \frac{d^2(rv)}{dr^2} + \left[k^2(r) - \frac{n(n+1)}{r^2}\right](rv) = 0 \quad .$$

We shall study this equation, not only for integral values of n which are the only ones needed to solve (14), but also for complex values of n which will be needed later. In case $k^2(r)$ is a constant, k_o^2 say, the solutions of (16) are arbitrary linear combinations of $r^{1/2}\, J_{n+1/2}(k_o r)$ and $r^{1/2}\, H^{(1)}_{n+1/2}(k_o r)$. The function $r^{1/2}\, J_{n+1/2}(k_o r)$ is regular at the origin if real part of $n > -1$ while the function $r^{1/2}\, H^{(1)}_{n+1/2}(k_o r)$ satisfies the radiation condition. If k^2 is not a constant, the solutions of (16) will behave approximately like the Bessel functions but their exact behavior will depend on the way k^2 varies with r.

We shall introduce three particular solutions of (16). Let $rv_n(r)$ be that solution which is regular at $r = 0$, while $rw_n^{(1)}(r)$ is that solution which for large r is asymptotically $e^{ik_o r}$, and $rw_n^{(2)}(r)$ is that solution which for large r is asymptotically $e^{-ik_o r}$. Note that the function $rw_n^{(1)}(r)$ satisfies the radiation condition (15). For simplicity of writing, we shall write it as $rw_n(r)$ when there is no possibility of confusion.

Since the differential equation (16) is unchanged when n is replaced by $-n - 1$, we see that

$$w_n^{(1)}(r) = w_{-n-1}^{(1)}(r) \quad , \quad w_n^{(2)}(r) = w_{-n-1}^{(2)}(r) \quad .$$

This relation is not true for $v_n(r)$ because the condition of regularity at the origin will imply that $v_n(r)$ behaves like $r^n (n > 0)$ at the origin and this condition is changed when n is replaced by $- n - 1$. It should also be noticed that for $k^2 = k_o^2$, a constant, while

$$rv_n(r) = r^{1/2} J_{n+1/2}(k_o r) \quad ,$$

on the other hand, because of the behavior at infinity, we have

$$rw_n^{(1)}(r) = \sqrt{\frac{\pi}{2}}\, e^{+i\pi/4} \quad e^{+i(2n+1)\pi/2}(k_o r)^{1/2}\, H_{n+1/2}^{(1)}(k_o r)$$

and

$$rw_n^{(2)}(r) = \sqrt{\frac{\pi}{2}}\, e^{-i\pi/4} \quad e^{-i(2n+1)\pi/2}(k_o r)^{1/2}\, H_{n+1/2}^{(2)}(k_o r) \quad .$$

We shall now prove the well-known fact that the Wronskian of any two solutions of (16) is a constant independent of r. Consider, for example, the two solutions $rv_n(r)$ and $rw_n(r)$. We have

$$\frac{d^2}{dr^2}(rv_n) + \left[k^2 - \frac{n(n+1)}{r^2}\right](rv_n) = 0$$

$$\frac{d^2}{dr^2}(rw_n) + \left[k^2 - \frac{n(n+1)}{r^2}\right](rw_n) = 0 \quad .$$

Multiply the first equation by rw_n, the second by rv_n and subtract. We get

$$(rw_n)\frac{d^2}{dr^2}(rv_n) - (rv_n)\frac{d^2}{dr^2}(rw_n) = 0 \quad .$$

Integrate this equation from $r = r_1$ to $r = r_2$, and we have

$$\left[rw_n(r)\frac{d}{dr}(rv_n) - rv_n(r)\frac{d}{dr}(rw_n)\right]_{r=r_2} = \left[rw_n(r)\frac{d}{dr}(rv_n) - rv_n(r)\frac{d}{dr}(rw_n)\right]_{r=r_1}$$

so that the Wronskian

$$W_n = rw_n(r)\frac{d}{dr}(rv_n) - rv_n(r)\frac{d}{dr}(rw_n) \tag{17}$$

is independent of the value of r but depends only on the value of n.

Now that we have some knowledge of the solutions of (16), we can prove that the solution of (14) is given by the following formulas:

$$
\begin{aligned}
ru_n(r) &= \frac{rv_n(r)w_n(b)}{2\pi\, W_n} \quad , \qquad r < b \\
&= \frac{rv_n(b)w_n(r)}{2\pi\, W_n} \quad , \qquad r > b \quad .
\end{aligned}
\tag{18}
$$

First, it is clear that both expressions in (18) satisfy (14) for $r \neq b$. Second, at $r = b$, the function ru_n is continuous and its derivative has a jump of magnitude $-1/2\pi b$. Third, from the definition of the functions $v_n(r)$ and $w_n(r)$, the function u_n is regular at $r = 0$ and satisfies the radiation condition (15). This shows that $ru_n(r)$ really is a solution of (14) satisfying all the other conditions.

Finally, we can obtain a representation for $U(r,\theta)$ from equation (12). We have

$$
\begin{aligned}
U(r,\theta) &= \frac{1}{2\pi} \sum_{0}^{\infty} \frac{v_n(r)w_n(b)}{W_n} \frac{2n+1}{2} P_n(\cos\theta) \quad , \quad r < b \\
&= \frac{1}{2\pi} \sum_{0}^{\infty} \frac{v_n(b)w_n(r)}{W_n} \frac{2n+1}{2} P_n(\cos\theta) \quad , \quad r > b \quad .
\end{aligned}
\tag{19}
$$

This is the solution of the problem in its most general form for an arbitrary variation of the dielectric constant. Special cases of this are well known for the cases in which the ordinary differential equation (16) can be solved in terms of known functions; for example, Rydbeck[3] has treated the problem where $\epsilon(r)$ is assumed to vary parabolically. It is to be noted that this solution (19) can be used for computation only if ka is small compared to unity. In the following sections this solution will be transformed so that it can be readily computed for large values of ka.

It is interesting to consider some special cases of (19). Suppose, first, that the medium is uniform and that there is no earth so that $k^2 = k_0^2$, a constant, then using the standard notation for the spherical Bessel function [13]

$$v_n(r) = j_n(k_o r)$$

$$w_n(r) = i^{n+1}\, h_n^{(1)}\,(k_o r)$$

$$W_n = i^n/k_o$$

formula (19) becomes

$$U(r,\theta) = \frac{e^{ik_oR}}{4\pi R} = + \frac{k_o i}{4\pi} \sum_0^\infty (2\,n+1)\, j_n(k_o r)\, h_n^{(1)}\,(k_o b)\, P_n(\cos\theta), \quad r < b$$

(20)

$$= + \frac{k_o i}{4\pi} \sum_0^\infty (2n+1)\, j_n(k_o b)\, h_n^{(1)}\,(k_o r)\, P_n(\cos\theta)\ , \quad r > b \quad ,$$

the classical formula for the dipole [13] , where R is the distance from the point with coordinates (r,θ) to the source point $(b,0)$. Suppose now that the earth is present so that for $r < a$, $k^2 = k_1^2$, $r > a$, $k^2 = k_o^2$: then

$$v_n(r) = j_n(k_1 r) \quad , \qquad r < a$$

$$v_n(r) = \gamma_n\, j_n\,(k_o r) + \delta_n\, h_n^{(1)}\,(k_o r) \quad , \qquad r > a$$

$$w_n(r) = i^{n+1}\, h_n^{(1)}\,(k_o r) \quad , \qquad r > a$$

where γ_n, δ_n are to be determined by the fact that $u_n(r)$ and $\frac{d}{dr} u_n(r)$ are continuous at $r = a$. From the definition (17) of the Wronskian, it follows that

$$W_n = i^{n+1}\, b^2 \gamma_n\, k_o \left[h_n^{(1)}\,(k_o b)\, j_n'\,(k_o b) - h_n^{(1)'}\,(k_o b)\, j_n\,(k_o b) \right]$$

$$= i^n\, \gamma_n / k_o \qquad .$$

γ is found from the following equations:

$$j_n(k_1 a) = \gamma_n \, j_n(k_0 a) + \delta_n \, h_n^{(1)}(k_0 a)$$

$$k_1 \, j_n'(k_1 a) = \gamma_n \, k_0 \, j_n'(k_0 a) + \delta_n \, k_0 \, h_n^{(1)'}(k_0 a) \quad .$$

We have

$$\gamma_n = \frac{k_1 \, j_n'(k_1 a) \, h_n^{(1)}(k_0 a) - k_0 \, j_n(k_1 a) \, h_n^{(1)}(k_0 a)}{h_n^{(1)}(k_0 a) \, j_n'(k_0 a) - h_n^{(1)'}(k_0 a) \, j_n(k_0 a)}$$

so that

$$W_n = i^{n+1} k_0 a^2 \left[k_1 \, j_n'(k_1 a) \, h_n^{(1)}(k_0 a) - k_0 \, j_n(k_1 a) \, h_n^{(1)}(k_0 a) \right] \tag{21}$$

Finally,

$$U(r,\theta) = \frac{1}{4\pi} \sum_{0}^{\infty} (2n+1) \frac{j_n(k_1 r) h_n^{(1)}(k_0 b) i^{n+1} P_n(\cos\theta)}{W_n} \quad , \quad r < a$$

$$= \frac{1}{4\pi} \sum_{0}^{\infty} (2n+1) \frac{\gamma_n j_n(k_0 r) + \delta_n h_n^{(1)}(k_0 r)}{W_n} \, i^{n+1} \, h_n^{(1)}(k_0 b) \, P_n(\cos\theta), \; a < r < b \tag{22}$$

$$= \frac{1}{4\pi} \sum_{0}^{\infty} (2n+1) \frac{\gamma_n j_n(k_0 b) + \delta_n h_n^{(1)}(k_0 b)}{W_n} \, i^{n+1} \, h_n^{(1)}(k_0 r) P_n(\cos\theta), \; r > b \quad .$$

Notice that

$$\gamma_n / W_n = k_0 i^{-n} ,$$

so that by using (20), (22) can be written as the sum of two terms:

$$U(r,\theta) = \frac{e^{ik_oR}}{4\pi r}$$

(23)

$$+ \frac{1}{4\pi} \sum (2n+1) \frac{\delta_n h_n^{(1)}(k_ob)}{W_n} i^{n+1} h_n^{(1)}(k_or)P_n(\cos\theta), \; r < b.$$

The first term is due to the dipole while the second term is due to the scattering of the dipole field by the earth.

In case the earth is assumed to be a perfect conductor so that $k_1 = \infty$, then $j_n(k_1a) = 0$ and from the continuity condition

$$\delta_n = - \gamma_n \, j_n(k_oa)/h_n^{(1)}(k_oa) \quad .$$

We find then that

$$U(r,\theta) = \frac{ik_o}{4\pi} \sum (2n+1) \left[j_n(k_or)h_n^{(1)}(k_oa)-h_n^{(1)}(k_or)\, j_n(k_or)\right] \frac{R_n^{(1)}(k_ob)}{h_n^{(1)}(k_oa)} P_n(\cos\theta), \; a<r<b$$

(24)

$$= \frac{ik_o}{4\pi} \sum (2n+1) \left[j_n(k_ob)h_n^{(1)}(k_oa)- h_n^{(1)}(k_ob)j_n(k_oa)\right] \frac{h_n^{(1)}(k_or)}{h_n^{(1)}(k_oa)} P_n(\cos\theta) \; r > b \quad .$$

4. Contour Integral Representation for U(r,θ)

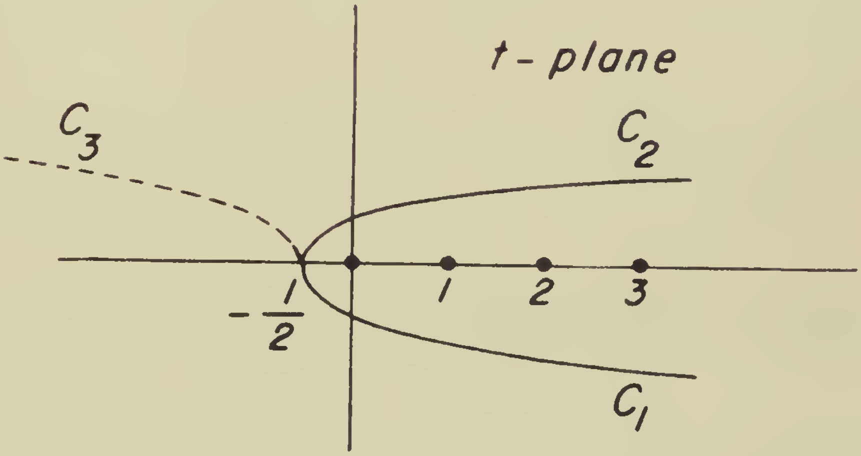
Following a method due to Watson [1], we shall show that the series (19) can be expressed as a contour integral. Consider the series

$$S = \sum_{0}^{\infty} \frac{2n+1}{2} a_n P_n (\cos \theta) \tag{25}$$

where a_n is assumed to be an analytic function of n. We shall show that S can be represented by the following integral:

$$I = \frac{1}{2\pi i} \int_C \frac{(t+1/2)\, a_t P_t (-\cos \theta)}{\sin \pi t} dt \tag{26}$$

where $C = C_1 + C_2$ is a contour that starts at $\infty - i\delta$ in the t-plane, goes below the real axis to $t = -1/2$ and then above the real axis to $\infty + i\delta$.



Since a_t and P_t $(-\cos\theta)$ are analytic functions of t, the only singularities of the integrand are poles at those values of t inside C for which $\sin \pi t = 0$, that is,

$$t = 0,\ 1,\ 2,\ \ldots$$

Since the contour C is described in the clockwise direction, the integral I is equal to the negative sum of the residues at the poles, so that

$$(27) \qquad I = -\sum_{0}^{\infty} \frac{(n+1/2)a_n P_n(-\cos\theta)}{\pi \cos \pi n} = -\frac{1}{\pi}\sum_{0}^{\infty} (n+1/2)\, a_n P_n\,(\cos\theta)$$

because

$$P_n(-\cos\theta) = (-1)^n P_n\,(\cos\theta) \qquad .$$

Formula (27) shows that

$$S = -\pi I \quad .$$

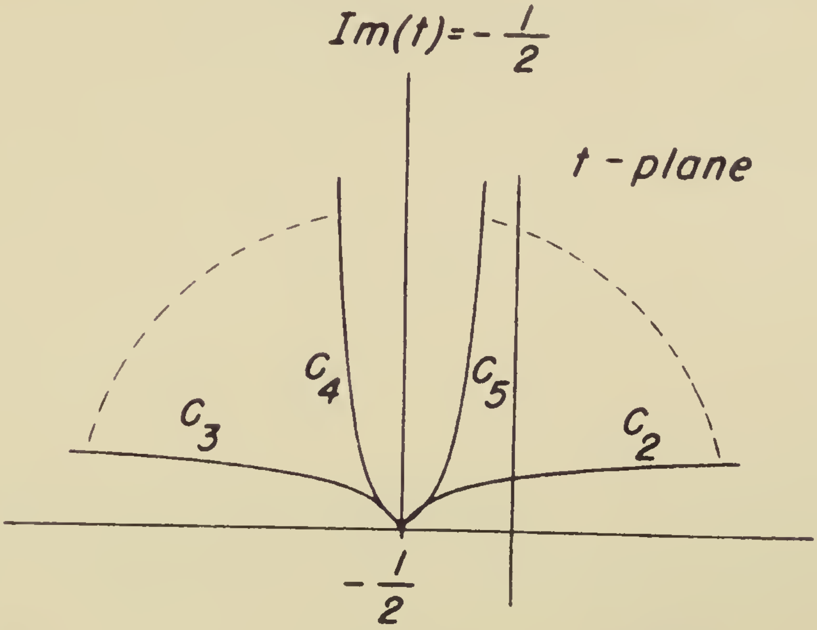
If this procedure is applied to (19), we find that

$$(28) \qquad U(r,\theta) = -\frac{1}{4\pi i}\int_C \frac{(t+1/2)v_t(b)w_t(r)}{W_t}\ \frac{P_t(-\cos\theta)}{\sin \pi t}\ dt$$

where C is the contour specified above. On C_1, the part of C below the real axis, replace t by $-t-1$, then C_1 is transformed into C_3 and we have

$$(29) \qquad \begin{aligned} U(r,\theta) = & -\frac{1}{4\pi i}\int_{C_3} \frac{(t+1/2)v_{-t-1}(b)w_{-t-1}(r)}{W_{-t-1}}\ \frac{P_{-t-1}(-\cos\theta)}{\sin \pi t}\ dt \\ & -\frac{1}{4\pi i}\int_{C_2} \frac{(t+1/2)v_t(b)w_t(r)}{W_t}\ \frac{P_t(-\cos\theta)}{\sin \pi t}\ dt \quad . \end{aligned}$$

We shall rotate C_2 and C_3 so that they go around the upper half of the line, imaginary part of $t = -1/2$. To justify this, it is necessary to show that the integrals over



the infinite quarter-circles in the first and second quadrants go to zero. This is discussed in the appendix.

When the contour is rotated in the desired manner, it may pass across some poles of the integrand. We find then that $U(r,\theta)$ as given by (29) can be written as the sum of residues at the poles plus two integrals over the upper half of the line $Im(t) = -1/2$. Put $t = -1/2 + i\tau$ and use the fact [14] that

$$(30) \qquad P_t(-\cos\theta) = P_{-t-1}(-\cos\theta) \quad ,$$

we obtain

$$(31) \quad U(r,\theta) = \sum \text{residues} + \frac{1}{4\pi i}\int_0^\infty \frac{\tau P_{i\tau-1/2}(-\cos\theta)}{-\cosh\pi\tau}\left[\frac{v_{i\tau-1/2}(b)w_{i\tau-1/2}(r)}{W_{i\tau-1/2}} - \frac{v_{-i\tau-1/2}(b)w_{-i\tau-1/2}(r)}{W_{-i\tau-1/2}}\right]d\tau \; .$$

From (28) it is clear that the poles of the integrand will be at the points where

$$W_t = 0 \quad .$$

Call these points t_1, t_2, ... ; then the residue at any pole, t_j, say, is

$$\frac{(t_j+1/2)v_{t_j}(b)w_{t_j}(r)P_{t_j}(-\cos\theta)}{W'_{t_j}\sin\pi t_j} \tag{32}$$

Here.

$$W'_{t_j} = \frac{d}{dt} W_t \Big|_{t=t_j} . \tag{33}$$

We thus have the final result:

$$U(r,\theta) = \sum_j \frac{(t_j+1/2)v_{t_j}(b)w_{t_j}(r)P_{t_j}(-\cos\theta)}{W'_{t_j}\sin\pi t_j}$$

$$+ \frac{1}{4\pi i}\int_0^\infty \frac{\tau P_{i\tau-1/2}(-\cos\theta)}{-\cosh\pi\tau}\left[\frac{v_{i\tau-1/2}(b)w_{i\tau-1/2}(r)}{W_{i\tau-1/2}} - \frac{v_{-i\tau-1/2}(b)w_{-i\tau-1/2}(r)}{W_{-i\tau-1/2}}\right] d\tau .$$

5. The Eigenvalue Problem.

The residue in (32) can be more usefully interpreted in another way. Since the Wronskian,

$$W_{t_j} = 0 ,$$

the two functions $v_{t_j}(r)$ and $w_{t_j}(r)$ are not independent; that is, there exist constants α_j such that

$$v_{t_j}(r) = \alpha_j w_{t_j}(r) .$$

This equation shows that the function $w_{t_j}(r)$, besides satisfying the radiation condition at infinity, also satisfies the condition of regularity at $r = 0$. In other words, t_j is an eigenvalue and $w_{t_j}(r)$ is an eigenfunction of the following problem:

Find the values of t for which $v(r)$, the solution of the following differential equation:

$$\frac{d^2(rv)}{dr^2} + \left[k^2 - \frac{t(t+1)}{r^2}\right] (rv) = 0 , \tag{34}$$

is regular at $r = 0$ and satisfies the radiation condition (15) at infinity, that is,

$$\lim_{r\to\infty} \left| \frac{d(rv)}{dr} - ik_0 rv \right| = 0 .$$

Put

$$\beta_j = \int_0^\infty w_{t_j}(r)^2 \, dr \tag{35}$$

so that $w_{t_j}(r)\beta_j^{1/2}$ is a normalized eigenfunction of (34); then the residue given by (32) will be shown to be equal to

(36) $$\frac{w_{t_j}(b)\, w_{t_j}(r)\, P_{t_j}(-\cos\theta)}{2\beta_j \quad \sin \pi t_j} .$$

This result will be obtained from a consideration of the following two equations:

$$\frac{d^2(rv_t)}{dr^2} + \left[k^2 - \frac{t(t+1)}{r^2}\right](rv_t) = 0$$

$$\frac{d^2(rw_{t_j})}{dr^2} + \left[k^2 - \frac{t_j(t_j+1)}{r^2}\right](rw_{t_j}) = 0 .$$

Here t has any value and is not an eigenvalue. Multiply the first equation by rw_{t_j}, the second by rv_t subtract and integrate with respect to r from 0 to ∞.

We obtain

(37) $$(t-t_j)(t+t_j+1)\int_0^\infty v_t\, w_{t_j}\, dr = rv_t \frac{d}{dr}(rw_{t_j}) - rw_{t_j} \frac{d}{dr}(rv_t)\Big|_0^\infty .$$

Notice the term at zero drops out because by definition v_t and w_{t_j} are regular at r = 0.

Consider the following determinant:

$$\begin{vmatrix} rv_t & rw_t & rw_{t_j} \\ rv_t & rw_t & rw_{t_j} \\ \frac{d}{dr}(rv_t) & \frac{d}{dr}(rw_t) & \frac{d}{dr}(rw_{t_j}) \end{vmatrix} .$$

It is obviously zero since the elements of the first two rows are identical. When it is expanded in terms of the elements of the first row, we have

$$rv_t\left[rw_t \frac{d}{dr}(rw_{t_j}) - rw_{t_j}\frac{d}{dr}(rw_t)\right] + rw_{t_j}\left[rv_t \frac{d}{dr}(rw_t) - rw_t \frac{d}{dr}(rv_t)\right]$$

$$- rw_t\left[rw_{t_j}\frac{d}{dr}(rv_t) - rv_t \frac{d}{dr}(rw_{t_j})\right] = 0 .$$

Now, as r approaches infinity, the first bracket approaches zero since rw_t and rw_{t_j} both satisfy the same condition at infinity. This shows that

$$\left[rv_t \frac{d}{dr}(rw_{t_j}) - rw_{t_j} \frac{d}{dr}(rv_t)\right] = \frac{w_{t_j}\left[rw_t \frac{d}{dr}(rv_t) - rv_t \frac{d}{dr}(rw_t)\right]}{w_t}$$

so that (37) can be written as follows:

$$\int_0^\infty v_t w_{t_j} \, dr = \frac{w_{t_j}(r)}{w_t(r)} \; \frac{W_t}{(t-t_j)(t+t_j+1)} \quad .$$

As t approaches t_j, w_t approaches w_{t_j} and v_t approaches $\alpha_j w_{t_j}$ while the right side of the above equation becomes indeterminate. If the right side is evaluated by L'Hôpital's rule, we find that

$$\alpha_j \int_0^\infty w_{t_j}(r)^2 \, dr = \frac{W'_t}{2t_j+1}$$

so that

$$\beta_j = \frac{W'_t}{(2t_j+1)\alpha_j}$$

and using this in (32) we get (36) .

When (36) is substituted in (31) we get finally the result that

$$(38) \qquad U(r,\theta) = \sum \frac{w_{t_j}(b) w_{t_j}(r)}{2\beta_j} \; \frac{P_{t_j}(-\cos\theta)}{\sin \pi t_j} + I$$

where I is the following integral:

$$I = \frac{1}{4\pi i}\int_0^\infty \frac{\tau P_{i\tau-1/2}(-\cos\theta)}{\cosh\pi\tau}\left[\frac{v_{-i\tau-1/2}(b)\,w_{-i\tau-1/2}(r)}{W_{-i\tau-1/2}} - \frac{v_{i\tau-1/2}(b)\,w_{i\tau-1/2}(r)}{W_{i\tau-1/2}}\right] d\tau \quad .$$

In the appendix it will be shown that, if the earth is assumed to be a perfect conductor, then I vanishes identically. If the earth is not assumed to be a perfect conductor, I will not be zero. However, it will be shown that I is small compared to the first term of the series in (38) as long as the dielectric constant of the earth is large compared to the dielectric constant of the atmosphere.

6. A Simpler Form for the Hertz Potential.

So far, we have not yet made use of the fact thatthe dielectric constant of the earth is a constant, nor of the fact that, for the wavelengths we are interested in, ka is very large, of the order of magnitude of 10^6. We shall use those facts to simplify the expansion given in (38).

It will be seen in the later sections that the zeros of the Wronskian, t_j, will all be very large, of the order of magnitude of ka. We are therefore justified in using an asymptotic formula for the Legendre functions that appear in (38). We have [14]

$$P_\nu\ (\cos\theta) \sim \left(\frac{2}{\pi(\nu+1)\sin\theta}\right)^{1/2} \cos\left\{\left(\nu+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right\}$$

for large values of ν, so that

$$\frac{P_t(-\cos\theta)}{\sin\pi\ t_j} = \frac{P_{t_j}(\cos(\pi-\theta))}{\sin\pi\ t_j} \sim -\left(\frac{2i}{\pi(t_j+1)\sin\theta}\right)^{1/2} e^{i(t_j+1/2)\theta} \quad . \tag{40}$$

Next, since for $r < a$, $k^2 = k_1^2$ = constant, we know that the solutions of (34) are

$$rv = r^{1/2}\ J_{t+1/2}\ (k_1 r) \quad \text{or} \quad r^{1/2}\ H^{(1)}_{t+1/2}\ (k_1 r), \quad r < a \quad .$$

The function $v_t(r)$ was defined to be that solution of (34) which was regular at $r = 0$. It must, therefore, be proportional to the function $J_{t+1/2}(k_1 r)$ for $r < a$. Since the solution of (34) and its first derivative must be continuous at $r = a$, this requires that

$$\left.\frac{\frac{d}{dr}\ rv_t(r)}{rv_t(r)}\right|_{r=a} = \left.\frac{\frac{d}{dr}\left(r^{1/2}\ J_{t+1/2}\ (k_1 r)\right)}{r^{1/2}\ J_{t+1/2}\ (k_1 r)}\right|_{r=a} \quad . \tag{41}$$

Again, the right side can be simplified by using the asymptotic formulas. Put

$$g(r) = \frac{\frac{d}{dr}\left(r^{1/2}\ J_{t+1/2}\ (k_1 r)\right)}{r^{1/2}\ J_{t+1/2}\ (k_1 r)}$$

so that

$$r^{1/2} J_{t+1/2}(k_1 r) = C e^{\int g(r)\, dr}$$

where C is a suitable constant. Substituting this expression in the differential equation for $r^{1/2} J_{t+1/2}(k_1 r)$, that is,

$$\frac{d^2(rv)}{dr^2} + \left[k_1^2 - \frac{t(t+1)}{r^2}\right](rv) = 0 ,$$

we get

$$\frac{dg}{dr} + g^2 + k_1^2 - \frac{t(t+1)}{r^2} = 0 . \tag{42}$$

Since $k_1^2 = \omega^2 \epsilon_1 \mu$, where ϵ_1, the dielectric constant of the earth, is very large, we may neglect $\frac{dg}{dr}$ in (42) and get the approximate formula:

$$g(r) = + i \sqrt{k_1^2 - \frac{t(t+1)}{r^2}} .$$

To determine whether it is a plus or minus sign, we notice that ϵ_1 and so, also k_1, has a large positive imaginary part so that, by the usual asymptotic formula,

$$\begin{aligned} r^{1/2} J_{t+1/2}(k_1 r) &\sim \sqrt{\frac{2}{\pi}} \cos\left\{k_1 r - \frac{\pi}{2}(t+1/2) - \frac{\pi}{4}\right\} \\ &\sim C_2 e^{-ik_1 r} \end{aligned}$$

where C_2 is independent of r. Then

$$\frac{\frac{d}{dr}\left(r^{1/2} J_{t+1/2}(k_1 r)\right)}{r^{1/2} J_{t+1/2}(k_1 r)} \sim -ik_1$$

and so $g(r)$, which is a more accurate expression for the ratio of the derivative to the function must have the minus sign, that is,

$$g(r) = -i\sqrt{k_1^2 - \frac{t(t+1)}{r^2}} \quad .$$

We shall see later that $t \sim ak(a)$. Because of this,

$$g(a) = -i\sqrt{k_1^2 - k^2} \quad ,$$

a quantity independent of t. Now (41) can be written as follows:

$$(43) \qquad \left.\frac{\frac{d}{dr}\, rv_t(r)}{rv_t(r)}\right|_{r=a} = -i\sqrt{k_1^2 - k^2} \quad .$$

This result holds for all values of t, if t is approximately $ak(a)$.

Consider the eigenvalues t_j for which the functions $v_t(r)$ and $w_t(r)$ coincide. Then (43) gives us the following eigenvalue equation:

$$(44) \qquad \left.\frac{\frac{d}{dr}\, rw_{t_j}(r)}{rw_{t_j}(r)}\right|_{r=a} = -i\sqrt{k_1^2 - k^2} \quad .$$

We shall discuss its solution in a later section.

There is one more quantity in the expansion (38) that will be simplified. It is the quantity β_j or essentially W_t'. Since the Wronskian is constant for all values of r, we have

$$W_t = rw_t \frac{d}{dr}(rv_t) - rv_t \frac{d}{dr}(rw_t)$$

and

$$W_t' = \frac{\partial(rw_t)}{\partial t}\frac{d}{dr}(rv_t) + rw_t \frac{\partial}{\partial t}\frac{d}{dr}(rv_t)$$

$$- \frac{\partial(rv_t)}{\partial t}\frac{d}{dr}(rw_t) - rv_t \frac{\partial}{\partial t}\frac{\partial}{\partial r}(rw_t) \quad .$$

all evaluated at $r = a$. Now, since the right side of (43) is independent of t, we have

$$\frac{\partial}{\partial t} \left. \frac{\frac{d}{dr}(rv_t)}{rv_t} \right|_{r=a} = 0$$

or

$$rv_t \frac{\partial}{\partial t} \left. \frac{d}{dr}(rv_t) \right|_a - \frac{\partial(rv_t)}{\partial t} \left. \frac{d}{dr}(rv_t) \right|_a = 0 \quad .$$

When $t = t_j$, an eigenvalue, the function $v_{t_j}(r)$ is identical with $w_{t_j}(r)$ and then

$$rw_{t_j} \frac{\partial}{\partial t} \frac{d}{dr}(rv_{t_j}) - \frac{\partial(rv_{t_j})}{\partial r} \frac{d}{dr}(rv_{t_j}) = 0 \quad .$$

Using this result in the formula for W_t', we find that

$$W_{t_j}' = \frac{\partial(rw_{t_j})}{\partial t} \left. \frac{d}{dr}(rw_{t_j}) \right|_a - rw_{t_j} \frac{\partial}{\partial t} \left. \frac{d}{dr}(rw_{t_j}) \right|_a \quad .$$

If $w_{t_j}(a) \neq 0$, which by (44) will always be the case if $k_1 \neq \infty$, that is, the earth is not a perfect conductor, then we may write the above formula as follows

$$W_{t_j}' = (rw_{t_j})^2 \frac{\partial}{\partial t} \left. \frac{\frac{d}{dr}(rw_{t_j})}{rw_{t_j}} \right|_a = (aw_{t_j})^2 M \quad , \tag{45}$$

where M is defined by the last equation.

Combining the results we have obtained in (45), (44), (42) and (38), we get, finally, the desired expression:

$$U(r,\theta) = - \sum_j \left[\frac{(2t_j+1)}{\pi \sin \theta}\right]^{1/2} e^{i(t_j+1/2)\,\theta} \; \frac{w_{t_j}(b) w_{t_j}(r)}{W'_{t_j}}$$

(46)

$$= \frac{-1}{br} \sum_j \left[\frac{(2t_j+1)i}{\pi \sin \theta}\right]^{1/2} \frac{e^{i(t_j+1/2)\theta}}{M} \; \frac{b w_{t_j}(b)}{a w_{t_j}(a)} \; \frac{r w_{t_j}(r)}{a w_{t_j}(a)} \quad .$$

Here, $b w_{t_j}(b)/a w_{t_j}(a)$ is the height gain factor for the transmitter and $r w_{t_j}(r)/a w_{t_j}(a)$ is the height gain factor for the receiver.

7. The Langer Asymptotic Formulas.

The previous sections have shown that the problem of determining the Hertz vector $U(r,\theta)$ and thus the electromagnetic field depends upon the solution of the eigenvalue problem for the differential equation (34). In general, this problem has to be solved by some approximation. In this section we shall discuss by a method due to Langer [9] the well-known W.K.B. approximation to the solutions of a second-order differential equation and in later sections we shall use these approximations to obtain the eigenvalues of (34), which we write as follows:

$$\frac{d^2(rv)}{dr^2} + \left[k^2(r) - \frac{\lambda}{r^2}\right](rv) = 0 . \tag{47}$$

Here, we have put λ for $t(t+1)$.

Consider a second order differential equation of the form:

$$\frac{d^2 s}{dr^2} + \left[\nu^2 f(r) + g(r)\right] s = 0 \tag{48}$$

where ν is assumed to be a large constant and where

$$f(r) = (r-r_0)^{\alpha} f_0(r) \quad ,$$

α is a constant and $f_0(r_0) \neq 0$. Put

$$r-r_0 = h(\tau) \quad , \qquad s = \sigma\,(h'(\tau))^{1/2} \tag{49}$$

where $h(\tau)$ is so chosen that

$$h'(\tau)^2 f(r) = 1 \tag{50}$$

and $\tau = 0$ for $r = r_0$. Under the transformations (49) equation (48) becomes

$$\frac{d^2\sigma}{d\tau^2} + \left\{ \nu^2 + \frac{\alpha(\alpha+4)}{4(\alpha+2)^2\tau^2} + \psi(\tau) \right\} \sigma = 0 \tag{51}$$

where $\psi(\tau)$ is small for large values of ν . The solutions of (51) are, approximately, linear combinations of

(52) $$\tau^{1/2} H_{\gamma}^{(1)} (\nu\tau) \text{ and } \tau^{1/2} H_{\gamma}^{(2)} (\nu\tau)$$

where

$$\gamma = (\alpha + 2)^{-1} \quad .$$

We shall see later that, for our purposes, only the cases $\alpha = 1$ and $\alpha = 2$ are of interest. The case $\alpha = 1$ corresponds to the situation where the atmosphere does not have a duct or a temperature inversion. The case $\alpha = 2$ will correspond to the situation where the atmosphere has a duct.

In case $\alpha = 1$, we rewrite equation (47) as follows:

$$\frac{d^2(rv)}{dr^2} + k_o^2a^2\left[\left(\frac{k(r)}{ak_o}\right)^2 - \frac{\lambda/a^2}{(k_or)^2}\right](rv) = 0 \quad .$$

This equation is of the same form as (48) if we make k_oa correspond to ν and put

$$f(r) = \left(\frac{k(r)}{k_oa}\right)^2 - \frac{\lambda/a^2}{(k_or)^2} \; , \quad g(r) = 0 \quad .$$

To use the formulas (52) we must put from (50)

$$\frac{d\tau}{dr} = \frac{1}{h'(\tau)} = \left[f(r)\right]^{1/2}$$

or

(53) $$\tau = \frac{1}{a}\int_{r_o}^{r} \left[\left(\frac{k(r)}{k_o}\right)^2 - \frac{\lambda}{(k_or)^2}\right]^{1/2} dr \quad .$$

Here, r_o is taken to be a zero of $f(r)$ so that

$$r_o^2\, k^2(r_o) = \lambda \quad .$$

If we change also the dependent variable from $v(r)$ to $\sigma(\tau)$ by the transformation indicated in (49):

$$rv = \sigma\left[h'(\tau)\right]^{1/2} = \sigma\left[\left(\frac{k(r)}{k_o a}\right)^2 - \frac{\lambda/a^2}{(k_o r)^2}\right]^{-1/4} ,$$

then, according to (52), we have

$$rw_t^{(1,2)} = \tau^{1/2}\left[\left(\frac{k(r)}{k_o a}\right)^2 - \frac{\lambda/a^2}{(k_o r)^2}\right]^{-1/2} H_{1/3}^{(1,2)}(k_o a\tau) \quad .$$

In particular, the solution which satisfies the Sommerfeld radiation condition at infinity will be given by the formula:

$$rw_t = \tau^{1/2}\left[\left(\frac{k(r)}{k_o a}\right)^2 - \frac{\lambda/a^2}{(k_o r)^2}\right]^{-1/4} H_{1/3}^{(1)}(k_o a\tau) \quad . \tag{54}$$

It should be noticed that the WKB method and the Langer formulas are not valid if $f(r)$ in (48) is discontinuous. This means that formula (54) is not valid for $r < a$. However, we have seen already that the solution of (34) for $r < a$ behaves like the Bessel function $J_{t+1/2}(k_1 r)$. Using the approximate formula (54) and neglecting terms of order $(ka)^{-1}$, we obtain a simpler form for the eigenvalue equation (44). It becomes

$$\left[\left(\frac{k(a)}{k_o}\right)^2 - \frac{\lambda}{(k_o a)^2}\right]^{1/2} \frac{H_{1/3}^{(1)}(k_o a\tau_a)'}{H_{1/3}^{(1)}(k_o a\tau_a)} = -i\,\frac{k_1}{k_o}\sqrt{1-\left(\frac{k(a)}{k_1}\right)^2} \quad . \tag{55}$$

Here τ_a is the value of τ corresponding to $r = a$. Note, that, if the earth is assumed to be a perfect conductor so that $k_1 = \infty$, then equation (55) reduces to the following simple form:

$$H_{1/3}^{(1)}(k_o a\tau_a) = 0 \quad . \tag{56}$$

Since the ratio

$$\frac{dH^{(1)}_{1/3}(x)}{dx} \Big/ H^{(1)}_{1/3}(x)$$

will appear so frequently in our analysis, we shall use for it the simpler notation Z(x). Equation (55) can now be written as follows:

$$\left[\left(\frac{k(a)}{k_o}\right)^2 - \frac{\lambda}{(k_o a)^2}\right]^{1/2} Z\,(k_o a\tau_a) = \frac{-ik_1}{k_o}\sqrt{1-\left(\frac{k(a)}{k_1}\right)^2} \quad .$$

Put

$$y = C_1\, H^{(1)}_{1/3}(x) = \exp\left[\int_{\alpha}^{x} Z(t)\, dt\right]$$

where C_1 and α are suitably chosen constants. From the differential equation satisfied by y

$$y'' + \frac{1}{x}y' + (1 - \frac{1}{9x^2})\, y = 0$$

we find that Z satisfies the Riccati equation:

$$Z' + Z^2 + \frac{1}{x} Z + 1 - \frac{1}{9x^2} = 0$$

so that for large x

(57) $$Z' \sim -1 - Z^2 - \frac{1}{x} Z \quad .$$

8. The Case of a Uniform Atmosphere.

We shall discuss first the case of a uniform atmosphere in order to more easily understand the case of a non-uniform atmosphere. The methods and results of this section are classical [11]. Suppose that the atmosphere is uniform so that $k^2 = k_o^2$ = constant and also that the earth is a perfect conductor. The solutions of

$$H^{(1)}_{1/3}(\zeta) = 0$$

are

(58) $$\zeta_o = 2.38\, e^{-\pi i}, \; \zeta_1 = 5.6\, e^{-\pi i}, \; \zeta_2 = 8.65\, e^{-\pi i}, \text{ etc.}$$

The equation (56) now becomes the following:

(59) $$k_o \int_{r_o}^{a} \left[1 - \frac{\lambda}{(k_o r)^2}\right]^{1/2} dr = \zeta_n = e^{-\pi i}\tau_n \;,$$

where ζ_n is defined by the last equation.

Put

(60) $$\rho = k_o r \lambda^{-1/2} \;, \rho_a = k_o a \lambda^{-1/2}$$

in (59). Since $k_o\, r_o = \lambda^{1/2}$, it reduces to this:

(60a) $$\lambda^{1/2} \int_1^{\rho_a} (1-\rho^{-2})^{1/2}\, d\rho = e^{-\pi i}\tau_n \;.$$

When we evaluate the integral we get, finally, the eigenvalue equation,

(61) $$(\rho_a^2 - 1)^{1/2} - \arctan (\rho_a^2 - 1)^{1/2} = e^{-\pi i}\, \tau_n \lambda^{-1/2} \;.$$

In this equation ρ_a is a function of λ defined by (60).

We shall find that $\lambda^{1/2}$ is of the order of magnitude of $(k_o a)$ and therefore the right side of (60) will be very near to zero which implies that ρ_a will be close to one.

Put

$$\rho_a = 1 + \delta (k_o a)^{-2/3} \;;$$

then (61) becomes

$$\frac{1}{3}(2\delta)^{3/2}\,(k_0\alpha)^{-1} = e^{-\pi i}\tau_n\,\lambda^{-1/2} = e^{-\pi i}\,\tau_n\,(k_0 a)^{-1}$$

or

$$\delta = \frac{1}{2}(3\tau_n)^{2/3}\,e^{-2\pi i/3}\;. \tag{62}$$

Each value of τ_n determines a value of δ, then a value of ρ_a, and by (60) a value of λ. Since t is approximately $\lambda^{1/2}$, we get finally the well-known expression [11] for the eigenvalue:

$$t_n = k_0 a + \frac{1}{2}(k_0 a)^{1/3}(3\tau_n)^{2/3}\,e^{\pi i/3}\;. \tag{63}$$

Note that (62) shows that δ is of the order of magnitude one and since $(k_0 a)^{-2/3}$ is small, our assumption that ρ_a is close to one is justified. It is also important to notice that the integral (60a) can be easily evaluated approximately. Since ρ_a is very close to one, we may proceed as follows:

$$\int_1^{\rho_a}(1-\rho^{-2})^{1/2}\,d\rho \sim \int_1^{\rho_a}(\rho^2-1)^{1/2}\,d\rho \sim 2^{1/2}\int_1^{\rho_a}(\rho-1)^{1/2}\,d\rho$$

$$= \frac{1}{3}\,2^{3/2}(\rho_a-1)^{3/2} = \frac{1}{3}(2\delta)^{3/2}(k_0 a)^{-1}$$

which is the same result that was obtained by an exact integration.

If we drop the assumption that the earth is a perfect conductor, then we must solve the eigenvalue equation (55) instead of the equation

$$H^{(1)}_{1/3}(\zeta) = 0\;.$$

Using the same approximations and notations as before, we find that

$$\left[1-\frac{\lambda}{(k_0 a)^2}\right]^{1/2} = \left[1-\frac{1}{\rho_a^2}\right]^{1/2} \sim \left[2(\rho_a-1)\right]^{1/2} = (2\delta)^{1/2}(k_0 a)^{-1/3}$$

and that

$$k_o a\tau = k_o \int_{r_o}^{a} \left[1 - \frac{\lambda}{(k_o r)^2}\right] dr = \lambda^{1/2} \int_{1}^{\rho_a} (1-\rho^{-2})^{1/2} d\rho \sim \frac{\Delta^{1/2}}{3} (k_o a)^{-1} (2\delta)^{3/2} \simeq \frac{1}{3} (2\delta)^{3/2}$$

so that (55) becomes the following:

$$(k_o a)^{-1/3} (2\delta)^{1/2} \, z\left\{(2\delta)^{3/2}/3\right\} = -i \frac{k_1}{k_o} \sqrt{1-\left(\frac{k(a)}{k_1}\right)^2} = \eta \quad , \tag{64}$$

where η is defined by the last equation. Methods for solving this equation are discussed by Bremmer [11].

We shall now find a simple approximate formula for the quantity M which appears in (45) and then we can write down the complete formula for the Hertz potential as given by (46). From (45), (57) and (64) we find that

$$M = \frac{\partial}{\partial t} \left. \frac{\frac{d}{dr}(rw_{t_j})}{rw_{t_j}} \right|_a = k_o (k_o a)^{-1/3} \frac{\partial}{\partial t} \left[(2\delta)^{1/2} z \left\{(2\delta)^{3/2}/3\right\}\right]$$

$$= k_o (k_o a)^{-1/3} \frac{\partial \delta}{\partial t} \frac{\partial}{\partial \delta} [\quad] = k_o (k_o a)^{-1/3} \frac{\partial \delta}{\partial t} \left[\frac{1}{(2\delta)^{1/2}} z + 2\delta z'\right]$$

$$= k_o (k_o a)^{-1/3} \frac{\partial \delta}{\partial t} \left[\frac{1}{(2\delta)^{1/2}} z - 2\delta \left\{1 + z^2 + \frac{3}{(2\delta)^{3/2}} z\right\}\right]$$

$$= k_o (k_o a)^{-1/3} \frac{\partial \delta}{\partial t} \left[2\delta + 2\delta z^2 + \frac{2}{(2\delta)^{1/2}} z\right]$$

$$= -k_o (k_o a)^{-1/3} \frac{\partial \delta}{\partial t} \left[2\delta + \eta^2 (k_o a)^{2/3} + \frac{2\eta (k_o a)^{1/3}}{2\delta}\right] \quad .$$

Now, by (60)

$$t(t+1) = \lambda = (k_o a)^2 \rho_a^{-2} = (k_o a)^2 \left[1 + \delta (k_o a)^{-2/3}\right]^{-2}$$

so that, approximately

$$t = k_0 a\left[1 + \delta (k_0 a)^{-2/3}\right]^{-1} = k_0 a - \delta (k_0 a)^{1/3}$$

and then

$$\frac{\partial \delta}{\partial t} = - (k_0 a)^{-1/3} .$$

Using this result in the expression for M given above, we find, finally, that

$$M = k_0 (k_0 a)^{-2/3} \left[2\delta + \eta^2 (k_0 a)^{2/3} + \eta (k_0 a)^{1/3} \delta^{-1}\right] . \tag{65}$$

Consider the case where both the transmitter and receiver are on the earth's surface so that b = a and r = a. Substituting (65) and (63) in (46) we get

$$U(a,\theta) = - \frac{e^{ik_0 a\theta}}{\pi a \theta} \sqrt{2\pi i (k_0 a)^{1/3}\theta} \sum_j \frac{\exp\left[i(k_0 a)^{1/3}\theta (3\tau_j)^{2/3}(e^{\pi i/3})/4\right]}{2\delta_j + \eta^2 (k_0 a)^{2/3} + \eta (k_0 a)^{1/3}\delta^{-1}} e^{i\theta/2} . \tag{66}$$

Here τ_j is the root of (64). Note that $a\theta = d$ is the distance from the transmitter to the receiver so that

$$U(a,\theta) = - \frac{e^{ik_0 d}}{\pi d} \sqrt{2\pi i \chi} \quad F(\chi)$$

where

$$\chi = (k_0 a)^{1/3} \theta$$

$$F(\chi) = \sum_j \frac{\exp\left[\chi (3\tau_j)^{2/3}(e^{\pi i/3})/4\right]}{2\delta_j + \eta^2 + \eta (k_0 a)^{1/3} \delta^{-1}} \quad e^{i\theta/2} .$$

As was pointed out before, these results are well known.

9. The Case of a Non-Uniform Atmosphere

If the atmosphere is not uniform and is such that $r^2k^2(r)$ has no stationary point, that is, physically, the atmosphere does not have a duct, then it can be treated similarly to the case of a uniform atmosphere. We shall see that the results of the preceding sections will be only slightly modified.

First, we find an approximate formula for τ as defined in (53). Since a will be near to r_0 and $\lambda^{1/2}$ to ak(a), the bracket in (53) may be approximated as follows. Put

$$k(a) = k_a \; ; \qquad \left[rk(r)\right]'_{r=a} = \zeta ; \qquad \zeta\, k_a\, k_0^{-2} = K^2$$

Then

$$\left(\frac{k(r)}{k_0}\right)^2 - \frac{\lambda}{(k_0 r)^2} \sim \frac{r^2k^2(r) - \lambda}{(k_0 a)^2} \sim \frac{2ak_a}{(k_0 a)^2}\left[rk(r) - \lambda^{1/2}\right]$$

(67)
$$\sim \frac{2ak_a}{(k_0 a)^2}\,\zeta\,(r - r_0) \sim 2\,\frac{k_a\,\zeta}{k_0^2}\,\left(\frac{r}{r_0} - 1\right) = 2K^2\left(\frac{r}{r_0} - 1\right) .$$

Here, we have made use of the fact that since $r_0 k(r_0) = \lambda^{1/2}$,

$$rk(r) - \lambda^{1/2} = \zeta\,(r - r_0)$$

approximately.

Now,

$$k_0 a\tau = 2^{1/2} k_0 K \int_{r_0}^{r} \left(\frac{r}{r_0} - 1\right)^{1/2} dr = 2^{3/2}\, K\, k_0 a \left(\frac{r}{r_0} - 1\right)^{3/2}/3 .$$

If we put, as before,

$$a = r_0\left[1 + \delta\,(Kk_0 a)^{-2/3}\right]$$

then

$$k_o a \tau_a = (2\delta)^{3/2}/3$$

and the eigenvalue equation (55) becomes the following:

$$(68) \qquad (Kk_o a)^{-1/3} (2\delta)^{1/2} Z\left\{(2\delta)^{3/2}/3\right\} = -i \frac{k_1}{Kk_o} \sqrt{1 - k_a^2 / k_1^2} = \bar{\eta} \quad .$$

where $\bar{\eta}$ is defined by the last equation.

Now

$$t \sim \lambda^{1/2} = r_o k(r_o) = ak_a + \zeta(r_o - a) \quad ,$$

by Taylor's Theorem, but

$$r_o - a = -\delta\, r_o\, (Kk_o a)^{-2/3} \sim -a\, \delta (Kk_o a)^{-1/3}$$

and so

$$(69) \qquad t = ak_a - \delta (Kk_o a)^{1/3} \frac{Kk_o}{k_a} \quad .$$

In most cases the term $ak'(a)$ will be negligible compared to k_a, so that $\zeta = k_a$ and $K = k_a/k_o$. Then

$$(70) \qquad t = ak_a - \delta (k_a a)^{1/3} \quad .$$

If this formula for t is compared with the corresponding formula (63) for the case of the uniform atmosphere, we observe that the only change is the substitution of k_a for k_o.

The value of M can be found by the method used in the preceding section. We have

$$N = Kk_o(Kk_oa)^{-1/3}\ \frac{\partial}{\partial t}\left[(2\delta)^{1/2}\, z\left\{(2\delta)^{3/2}/3\right\}\right]$$

$$= Kk_o(Kk_oa)^{-1/3}\ \frac{\partial\delta}{\partial t}\left[2\delta + 2\delta z^2 + \frac{2}{(2\delta)^{1/2}}\, z\right]$$

$$= -Kk_o(Kk_oa)^{-1/3}\ \frac{\partial\delta}{\partial t}\left[2\delta + \bar\eta^2(Kk_oa)^{2/3} + \bar\eta(Kk_oa)^{1/3}\delta^{-1}\right] .$$

Now, from (69),

$$\frac{\partial\delta}{\partial t} = -\frac{k_a}{Kk_o}(Kk_oa)^{-1/3}$$

so that

(71)
$$N = -k_a(Kk_oa)^{-2/3}\left[2\delta + \bar\eta^2(Kk_oa)^{2/3} + \bar\eta(Kk_oa)^{1/3}\delta^{-1}\right] .$$

Again, consider the case where the transmitter and receiver are both on the earth's surface so that b = r = a. Using (69) and (71) in (46), we have

$$U(a,\theta) = -\frac{\exp\left[ik_aa\theta\right]}{\pi a\theta}\ \frac{\sqrt{2\pi i(Kk_oa)^{1/3}\theta}}{k_a(Kk_o)^{-1}}\ \sum_j \frac{\exp\left[-i(Kk_oa)^{1/3}\theta Kk_o\,\delta/k_a\right]}{2\,\delta_j + \bar\eta^2(Kk_oa)^{2/3} + \bar\eta(Kk_oa)^{1/3}\delta^{-1}}\, e^{i\theta/2}$$

(72)
$$= -\frac{\exp(ik_ad)}{\pi a\theta}\ \frac{\sqrt{2\pi i\chi_1}}{k_a(Kk_o)^{-1}}\ F_1(\chi_1)$$

where

$$\chi_1 = (Kk_oa)^{1/3}\theta$$

(73)
$$F_1(\chi_1) = \sum_j \frac{\exp\left[-i\chi_1 Kk_o\,\delta/k_a\right]}{2\,\delta_j + \bar\eta^2(Kk_oa)^{2/3} + \bar\eta(Kk_oa)^{1/3}\delta^{-1}}\ e^{i\theta/2} .$$

Note that again, if $ak'(a)$ is negligible to k_a , so that $K = k_a/k_o$,

$$\chi_1 = (k_a a)^{1/3}\, \theta$$

$$F_1\,(\chi_1) = F\,(\chi)$$

as defined in the preceding section and

$$U(a,\theta) = - \frac{\exp(ik_a d)}{\pi\, a\, \theta} \sqrt{2\pi i \chi_1}\;\; F(\chi_1) \;. \tag{74}$$

This formula shows that the Hertz potential in the case of a non-uniform atmosphere can be obtained from the Hertz potential in the cases of a constant atmosphere by making the substitutions: k_a for k_o and $\bar{\eta}$ for η where $\bar{\eta}$ is obtained from η by replacing k_o by k_a.

The case where the transmitter and receiver are not on the earth's surface can be derived from the preceding results by multiplying each term in the series for $F(\chi)$ by the corresponding height-gain factors for the transmitter and receiver. The height-gain factor for the receiver is

$$\frac{r w_{t_j}(r)}{a w_{t_j}(a)} = \left(\frac{\tau}{\tau_a}\right)^{1/2} \left[\frac{(ak_a)^2 - \lambda}{(rk)^2 - \lambda}\right]^{1/4} \left(\frac{r}{a}\right)^{1/2} \frac{H^{(1)}_{1/3}(k_o a \tau)}{H^{(1)}_{1/3}(k_o a\, \tau_a)} \tag{75}$$

by (54). We shall introduce into this formula θ_r, the angular distance of the receiver from the horizon, and then (75) will take a simpler form.

From the definition of θ_r, and since $r - a$ is small compared to a, we have

$$\cos\theta_r = \frac{a}{r} \approx 1 - \frac{r-a}{a}$$

and since

$$\cos\theta_r = 1 - \theta_r^2/2 \;,$$

we have

$$\theta_r = \sqrt{2(r-a)/a} \quad .$$

Put

$$\chi_r = (ak_a)^{1/3}\,\theta_r \quad ;$$

then, from (67)

$$\left(\frac{k}{k_o}\right)^2 - \frac{\lambda}{(k_o r)^2} \sim 2K^2\left(\frac{r}{r_o} - 1\right) = 2K^2\left[\frac{r}{a}\left\{1+\delta\,(ak_a)^{-2/3}\right\} -1\right]$$

$$= K^2(ak_a)^{-2/3}\,(\chi_r^2 + 2\delta)$$

and we find that

$$k_o a\tau = (\chi_r^2 + 2\delta)^{3/2}/3 \quad .$$

Since for r = a, $\chi_r = 0$, we see that the right hand member of (75) becomes

$$(76) \qquad \left[\frac{\chi_r^2 + 2\delta_j}{2\delta_j}\right]^{1/2} \frac{H_{1/3}^{(1)}\left\{(\chi_r^2 + 2\delta)^{3/2}/3\right\}}{H_{1/3}^{(1)}\left\{(2\delta)^{3/2}/3\right\}} \quad .$$

There is a similar formula for the height gain factor for the transmitter. Instead of χ_r in (76) we must use χ_b where

$$\chi_b = (ak_a)^{1/3}\,\theta_b, \quad \theta_b = \sqrt{2(b-a)/a} \quad .$$

It should be pointed out that results similar to those of this section, which show that the field due to a non-uniform atmosphere could be obtained from the field in a uniform atmosphere, have been obtained by Eckersley [10] and Bremmer [11].

However, they had to assume a particular form of variation of the dielectric constant, to wit,

$$\epsilon(r) = 1 - \alpha + \beta a^2/r^2$$

where α and β are small constants. The results of this section are independent of the form of variation of the dielectric constant.

10. Connection with the "Flat Earth" Theory.

The preceding sections have shown that to a good order of accuracy the eigenvalues depend on the behavior of $k^2(r)$ only in the neighborhood of the transition point, that is, the point at which

$$k^2(r) - \frac{\lambda}{r^2} = 0 \quad .$$

It was also shown that at the transition point $r_0 = a$, very closely. If these facts are taken into account, the differential equation (34) may be approximated as follows:

$$\frac{d^2(rv)}{dr^2} + \left[k_a^2 N^2 - k_n^2 \right] (rv) = 0 \tag{77}$$

where

$$N^2 = \frac{r^2 k^2(r)}{a^2 k_a^2}$$

is the modified index of refraction and where

$$k_n^2 = t_n(t_n+1)/a^2 \quad .$$

Equation (77) is exactly the eigenvalue equation that is obtained in the troposphere theory on the assumption of a flat earth [15]. If the W.K.B. method is used to solve this equation, we see that we get the same results as by applying the W.K.B. method to the more exact equation (34). There will be a discrepancy between the two methods only at points where r differs very much from a. This indicates that the height-gain factors on the "flat earth" theory will be wrong for large heights above the earth's surface. Another possible source of error is due to the fact that the Legendre functions of the exact theory have been approximated by exponentials. This approximation will be bad for extremely large distances between receiver and transmitter. Similar conclusions on the range of validity of the "flat earth" theory were reached by Pekeris [7]. On the whole, however, the results of the previous sections have shown that the "flat earth" theory gives a very satisfactory approximation to the solution.

11. An Atmosphere Containing a Duct.

In Section 9 we have discussed propagation in a non-uniform atmosphere under the assumption that $r^2k^2(r)$ does not have a stationary point near $r = a$. The reason for this assumption is that, otherwise, the differential equation (34) will have two transition points, that is, the equation

$$k^2(r) - \frac{\lambda}{r^2} = 0$$

will have two roots in λ near $\lambda = (ak_a)^2$. Now, the Langer approximation by means of Bessel functions of order one-third is valid in the neighborhood of each transition point separately but it cannot be used in a neighborhood of both transition points. Langer [17] has pointed out that in such a case the approximation to the solutions of the differential equation should be in terms of confluent hypergeometric functions. Since these functions are not too well tabulated, we shall attempt another approach.

In practice, the two transition points of the differential equation are close to each other. We shall approximate the differential equation by assuming that the transition points coincide so that we have a double transition point. In such a case we have $\alpha = 2$ in equation (51) and the approximate solution will be expressed in terms of Bessel functions of order one-fourth.

Let r_o be a zero of

$$k^2(r) - \frac{\lambda}{r^2} = 0 .$$

Put

$$f(r) = \frac{k^2(r)}{k_o^2 a^2} - \frac{\lambda}{k_o^2 a^2 r^2} - \frac{(r-r_o)}{k_o a^2} \left[k^2(r) - \frac{\lambda}{r^2} \right]'_{r=r_o}$$

and

$$g(r) = (r-r_o) \left[k^2(r) - \frac{\lambda}{r^2} \right]'_{r=r_o}$$

so that $f(r)$ has a double zero at $r = r_o$. According to (50),

$$\tau = \int_{r_o}^{r} \left[f(r) \right]^{1/2} dr$$

and then from (52) since $\alpha = 2$, we have that

$$rw_t = \tau^{1/2}\left[f(r)\right]^{-1/4} H^{(1)}_{1/4}\,(k_o a \tau)\,. \tag{78}$$

The eigenvalue equation now becomes

$$af(a)^{1/2}\quad \frac{H^{(1)}_{1/4}\,(k_o a \tau_a)'}{H^{(1)}_{1/4}\,(k_o a \tau_a)} = -\,i\,\frac{k_1}{k_o}\sqrt{1-\left(\frac{k_a}{k_1}\right)^2}\quad . \tag{79}$$

In case the earth is assumed to be a perfect conductor, the equation reduces to

$$H^{(1)}_{1/4}\,(k_o a \tau_a) = 0\,.$$

The first two roots of the equation

$$H^{(1)}_{1/4}\,(\chi) = 0$$

are $\chi = (2.38 + 0.17i)\,e^{-\pi i}$, $\chi = (5.6 + 0.17i)\,e^{-\pi i}$, so that the eigenvalue equation becomes

$$k_o a \tau_a = (2.38 + 0.17i)\,e^{-\pi i}\,. \tag{80}$$

The integral for τ can be easily approximated in certain cases. If

$$\frac{f'''(r_o)}{f''(r_o)}\,(r - r_o) << 1$$

then we may write

$$f(r) \sim \frac{1}{2}\,f''(r_o)\,(r-r_o)^2\,.$$

Now

$$f''(r_0) = \left[\frac{(k^2)''}{k_0^2\, a^2} - \frac{6\lambda}{k_0^2\, a^2\, r^4} \right]_{r = r_0} = \frac{2\xi}{a^3} ,$$

where ξ is defined by the last equation, so that

$$f(r) = \frac{\xi}{a^3} (r - r_0)^2$$

and

$$(81) \qquad k_0 a \tau_a = k_0 a \int_{r_0}^{a} \frac{\xi^{1/2}}{a^{3/2}} (r - r_0)\, dr = k_0 a\, (\xi\, a)^{1/2} \left(\frac{a}{r_0} - 1 \right)^2 /2 .$$

Put, as before

$$a = r_0 (1 + \bar{\delta})$$

in equation (81), then

$$k_0 a \tau_a = \frac{1}{2} k_0 a\, (\xi\, a)^{1/2} \bar{\delta}^2$$

and in case the earth is a perfect conductor, we must solve the equation:

$$\frac{1}{2} (\xi\, a)^{1/2} \bar{\delta}^2 = (k_0 a)^{-1} (2.38 + 0.17i)\, e^{-\pi i} .$$

Note that ξ will depend on r_0 and therefore also on $\bar{\delta}$. In case $\xi^{1/2}$ is a slowly varying function of $\bar{\delta}$, we may consider it a constant and then

$$\bar{\delta} = (4.76 + 0.34i)^{1/2}\, e^{-\pi i/2} (k_0/\xi)^{1/2} (k_0 a)^{-3/4} .$$

Since

$$t \sim \lambda^{1/2} = k(r_0)\, r_0 \sim k_0\, r_0 \sim k_0\, a_0\, (1 - \delta)$$

we find that

$$(82) \qquad t \sim k_0 a + (k_0/\zeta)^{1/2} (k_0 a)^{1/4} (4.76 + 0.34i)^{1/2} e^{\pi i/2} \quad .$$

Formula (82) shows that the imaginary part of t varies as $(k_0 a)^{1/4}$ for the case of a duct, as contrasted to the case of a homogeneous atmosphere where it varied as $(k_0 a)^{1/3}$. Since the attemuation of the field in decibels depends on the imaginary part of t, we can see that the range will be greatly increased in the case of a duct. For $(k_0 a)$ of the order of 10^8, the range is multiplied by 4.6.

12. The Electric Dipole.

The preceding work has been carried out for the case of the magnetic dipole because then the equations and the discontinuity conditions become simpler. However, the case of most practical interest is that of an electric dipole. In this section we shall give briefly the equations appropriate for an electric dipole.

First, a new Hertz vector and potential must be defined. Instead of (2) we put

$$\underline{H} = \nabla \times \underline{C}_1^+ \tag{83}$$

where $\underline{C}_1^+$ again has only its radial component not zero so that

$$\underline{C}_1^+ = (rU_1,\ 0,\ 0)\ .$$

Instead of (6), we find

$$\epsilon \frac{\partial}{\partial r} \frac{1}{\epsilon} \frac{\partial (rU_1)}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial (rU_1)}{\partial \theta} + k^2 rU_1 = 0 \tag{84}$$

and instead of (8), we have

$$E_r = \frac{i}{\epsilon\omega}\left(\epsilon \frac{\partial}{\partial r} \frac{1}{\epsilon} \frac{\partial}{\partial r} + k^2\right)(rU_1)$$

$$H_\phi = -\frac{\partial U_1}{\partial \theta}\ , \quad E_\theta = -\frac{1}{\epsilon r} \frac{\partial^2}{\partial r \partial \theta}(rU_1)\ , \quad H_r = H_\theta = E_\phi = 0\ .$$

Since E_θ and H_ϕ must be continuous across a discontinuity surface r = constant, it follows that U_1 and $\frac{1}{\epsilon}\frac{\partial U_1}{\partial r}$ must be continuous across this surface. Instead of (14) we have the following equation:

$$\epsilon \frac{d}{dr} \frac{1}{\epsilon} \frac{d(ru_n)}{dr} + \left[k^2 - \frac{n(n+1)}{r^2}\right](ru_n) = -\frac{\partial(r-b)}{2\pi r} \tag{86}$$

with the same boundary conditions as for (14). Put again $v_n(r)$ for those solutions of the homogeneous equation corresponding to (86), which are regular at the origin, and $w_n^{(1)}(r)$ for those solutions of the homogeneous equation which satisfy the radiation condition at infinity. We find that, now, the Wronskian is no longer a constant, but that

(87) $$W_n(r)/\epsilon_n(r) = \text{constant}.$$

The formula for $U_1(r,\theta)$ is the same as (19):

$$U_1(r,\theta) = \frac{1}{2\pi}\sum_0^\infty \frac{v_n(r)w_n(b)}{W_n(b)}\,\frac{2n+1}{2}\,P_n(\cos\theta)\ , \qquad r < b$$

(88)

$$= \frac{1}{2\pi}\sum_0^\infty \frac{v_n(b)w_n(r)}{W_n(b)}\,\frac{2n+1}{2}\,P_n(\cos\theta)\ , \qquad r < b \quad .$$

The contour integral representation and all the transformations are unchanged. However, the eigenvalue problem becomes the following:

Find those values of t for which there exists a solution of the differential equation:

(89) $$\epsilon\,\frac{d}{dr}\,\frac{1}{\epsilon}\,\frac{d(rv)}{dr} + \left(k^2 - \frac{t(t+1)}{r^2}\right)(rv) = 0$$

which is both regular at $r = 0$ and satisfies the radiation condition (15) at infinity.

Because of the different discontinuity conditions, the eigenvalue equation is changed. Instead of (41) we have

$$\left.\frac{\frac{1}{\epsilon(r)}\,\frac{d}{dr}\,(rv_t)}{rv_t}\right|_{r=a} = \left.\frac{\frac{1}{\epsilon_1}\,\frac{d}{dr}\left\{r^{1/2}\,J_{t+1/2}(k_1 r)\right\}}{r^{1/2}\,J_{t+1/2}(k_1 r)}\right|_{r=a}$$

so that the eigenvalue equation (44) becomes

(90) $$\left.\frac{\frac{d}{dr}\,(rw_{t_j})}{rw_{t_j}}\right|_{r=a} = -\,\frac{\epsilon(a)}{\epsilon_1}\;i\sqrt{k_1^2 = k^2} \quad .$$

After transforming the contour integral for (88) and using the asymptotic formulas for the Legendre functions, we get the following result, instead of (46)

$$(91) \qquad U(r,\theta) = - \sum_j \left[\frac{(2t_j+1)i}{\pi \sin\theta}\right]^{1/2} \frac{e^{i(t_j+1/2)\theta}}{\left.\frac{\epsilon(b)}{\epsilon(a)} \frac{\partial}{\partial t} \frac{\frac{d}{dr}(rw_{t_j})}{rw_{t_j}}\right|_a} \frac{w_{t_j}(b)}{aw_{t_j}(a)} \frac{w_{t_j}(r)}{aw_{t_j}(a)} .$$

Here $t_j (j = 1, 2, \ldots)$ are the eigenvalues of (89).

Equation (89) can be treated by the W.K.B. method after a transformation is made, to get rid of the ϵ^{-1} factor. Put

$$rv = \epsilon^{1/2} r\bar{v}$$

then equation (89) becomes

$$(92) \qquad \frac{d^2}{dr^2}(r\bar{v}) + \left(\bar{k}^2 - \frac{t(t+1)}{r^2}\right)(r\bar{v}) = 0$$

where

$$\bar{k}^2 = k^2 + \frac{1}{2}\epsilon^{1/2}(\epsilon^{-3/2}\epsilon')' = k^2 - k\frac{d^2}{dr^2}(k^{-1}) .$$

This transformation is valid only if we assume ϵ is continuous and differentiable. Since we shall only apply the W.K.B. method to (89) for $r > a$, our assumption is justifiable.

The eigenvalues at t_j of (92) can be found in exactly the same way as we found the eigenvalues for (34). We shall not repeat the work but one difference should be noted. In case the earth is assumed to be a perfect conductor, so $\epsilon_1 = k_1 = \infty$, the eigenvalue equation (90) becomes

$$\left.\frac{\frac{d}{dr}(rw_{t_j})}{rw_{t_j}}\right|_{r=a} = 0$$

or

$$\left.\frac{d}{dr}(rw_{t_j})\right|_{r=a} = 0 .$$

instead of

$$rw_{t_j} = 0 .$$

Appendix I - The Contour Integral on the Semicircle.

In this section we complete the discussion of the contour integral representation in (29) by showing that the integrals over the semi-circle in the first and second quadrant vanishes as the radius of the circle goes to infinity. This justifies the expansion of the integral as a sum of residues and an integral over the imaginary axis.

Consider, now, the integral

$$(93) \qquad \int \frac{(t + 1/2)v_t(b)w_t(r)}{W_t} \quad \frac{P_t(-\cos \theta)}{\sin \pi t} \quad d\,t$$

over a quarter circle $|t| = T$ in the first quadrant. Put $t = T e^{i\theta}$, then from (40) we have

$$\left| \frac{P_t(-\cos \theta)}{\sin \pi t} \right| \leq \frac{e^{-T\theta \sin \theta}}{\theta T^{1/2}} \quad .$$

To estimate the other factor in the integrand, we shall investigate the asymptotic behavior for large t of the solutions of the equation corresponding to (16):

$$\frac{d^2(rv)}{dr^2} + (k^2 - \frac{t(t+1)}{r^2})\,(rv) = 0 \quad .$$

It can be easily verified by substitution in the differential equation that there are two independent solutions which behave for large t either like

$$rv \sim C_1 r^{t+1} \left[1 - \frac{1}{2t+1} \int_0^r k^2 r \right] dr$$

or

$$rv \sim C_2 r^{-t} \left[1 + \frac{1}{2t} \int_0^r k^2 r \quad dr\right]$$

where C_1 and C_2 are independent of r. Since rv_t was by definition regular at $r = 0$, it must behave like $C_1 r^{t+1}$ for large t. When these asymptotic formulas are differentiated with respect to t, we find that for all solutions of the differential equations

(94) $$\left| \frac{\frac{d}{dr}(rv)}{rv} \right| < \frac{t}{r} .$$

Now

$$\frac{(t+1/2)v_t(b)w_t(r)}{W_t} = \frac{(t+1/2)bv_t(b)rw_t(r)}{br\left[rw_t(r)\,\frac{d}{dr}\,(rv_t(r)-rv_t(r)\,\frac{d}{dr}\,(rw_t(r))\right]}$$

$$= \frac{t+1/2}{br}\,\frac{\frac{bv_t(b)}{rv_t(r)}}{M}$$

where

$$M = \frac{\frac{d}{dt}(rv_t)}{rv_t} - \frac{\frac{d}{dt}(rw_t)}{rw_t} .$$

M will vanish if and only if $W_t = 0$. If T is so chosen that the quarter circle does not go through any zero of W_t, then from(94) M will be bounded below by some multiple of t/r. We have then

$$\frac{(t+1/2)v_t(b)w_t(r)}{W_t} \sim \frac{Ct}{br}\left(\frac{b}{r}\right)^{t+1} \frac{1}{t/r} = \frac{C}{b}\left(\frac{b}{r}\right)^{t+1}$$

and since $r < b$, this term will go to zero as T goes to infinity. Finally, the integral will be less in absolute value then

$$\frac{C}{b\theta T^{1/2}} \int_0^{\pi_2} e^{-T\theta \sin \theta} \quad T d\theta \left(\frac{b}{r}\right)^{T \cos \theta}$$

and as this clearly approaches zero for large T, the shifting of the contour in section 4 was justified.

Appendix II. An Estimate for the Integral.

It has been shown that the contour integral representation of the Hertz potential can be expressed (39) as a sum of residues plus an integral over the imaginary axis. In this section we estimate the value of the integral and show two things:

First, the integral vanishes identically if the earth is assumed to be a perfect conductor; second, the integral is small compared to the first term in the residue series if the complex dielectric constant of the earth is small compared to the dielectric constant of the atmosphere. Since this condition is always satisfied in practice, we are justified in neglecting the contribution of the integral.

In order to estimate the integrand in (39) we must make use of the fact that the differential equation (34) is unchanged when t is replaced by -t-1 so that

$$w_t^{(1)}(r) = w_{-t-1}^{(1)}(r) \qquad \text{and} \qquad w_t^{(2)}(r) = w_{-t-1}^{(2)}(r) \ .$$

Since $w_t^{(1)}$ and $w_t^{(2)}$ are independent solutions of (34), we may write

$$v_t(r) = \alpha\, w_t^{(1)}(r) + \beta\, w_t^{(2)}(r)$$

where α and β are independent of r but may depend on t.

Note that

$$w_t = \beta \text{ Wronskian } (w_t^{(2)}, w_t^{(1)}) = 2ik_0\beta \ .$$

Now, we make use of the fact that for $r < a$, $k^2 = k_1^2$, a constant, so that

$$r v_t(r) = r^{1/2} J_{t+1/2}(k_1 r) \qquad r < a \ .$$

At the boundary $r = a$, we must have

$$\alpha\, a w_t^{(1)}(a) + \beta\, a w_t^{(2)}(a) = a^{1/2}\, T_{t+1/2}(k_1 a)$$

$$\alpha \frac{d}{dr}(r w_t^{(1)})\Big|_a + \beta \frac{d}{dr}(r w_t^{(2)})\Big|_a = \frac{d}{dr}(r^{1/2} J_{t+1/2})\Big|_a$$

These two equations will determine α and β. We find

$$\frac{\alpha}{\beta} = - \left. \frac{rw_t^{(2)} \frac{d}{dr}\left(r^{1/2} J_{t+1/2}\right) - r^{1/2} J_{t+1/2} \frac{d}{dr} (rw_t^{(2)})}{rw_t^{(1)} \frac{d}{dr}\left(r^{1/2} J_{t+1/2}\right) - r^{1/2} J_{t+1/2} \frac{d}{dr} (rw_t^{(1)})} \right|_{r=a}$$

$$= - \frac{rw_t^{(2)}(r)}{rw_t^{(1)}(r)} \left. \frac{\frac{d}{dr}\left[\log (r^{1/2} J_{t+1/2})\right] - \frac{d}{dr}\left[\log (rw_t^{(2)})\right]}{\frac{d}{dr}\left[\log r^{1/2} J_{t+1/2}\right] - \frac{d}{dr}\left[\log (rw_t^{(1)})\right]} \right|_{r=2} \tag{95}$$

$$= - \frac{aw_t^{(2)}(a)}{aw_t^{(1)}(a)} M_t \, ,$$ when M_t is defined by the last equation.

Now, consider the expression

$$\frac{v_t(b)\, w_t(r)}{W_t} = \frac{\left[\alpha w_t^{(1)}(b) + \beta w_t^{(2)}(b)\right] w_t^{(1)}/(r)}{2\, ik_o \beta}$$

$$= \frac{w_t^{(2)}(b) w_t^{(1)}(r)}{2ik_o} - \frac{w_t^{(1)}(a)}{2ik_o w_t^{(1)}(a)} w_t^{(1)}(b) w_t^{(1)}(r)\, M_t \, .$$

Everything in this formula, except the factor M_t, is unchanged when t is replaced by -t-1, so that

$$\frac{v_t(b) w_t(r)}{W_t} - \frac{v_{-t-1}(b)\, w_{-t-1}(r)}{W_{-t-1}} = - \frac{w_t^{(2)}(a) w_t^{(1)}(b) w_t^{(1)}(r)}{2ik_o w_t^{(1)}(a)} \left[M_t - M_{-t-1}\right] .$$

Now

$$M_t = \frac{\left[\frac{d}{dr} \log (r^{1/2} J_{t+1/2})\right]_a - \left[\frac{d}{dr} \log rw_t^{(2)}\right]_a}{\left[\frac{d}{dr} \log (r^{1/2} J_{t+1/2})\right]_a - \left[\frac{d}{dr} \log rw_t^{(1)}\right]_a}$$

$$= \frac{\left[\frac{d}{dr} \log rw_t^{(1)}\right]_a - \left[\frac{d}{dr} \log rw_t^{(2)}\right]_a}{\left[\frac{d}{dr} \log (r^{1/2} J_{t+1/2})\right]_a - \left[\frac{d}{dr} \log rw_t^{(1)}\right]_a} + 1 \tag{96}$$

and we see that in this formula the only term which changes when t is replaced by $-t-1$ is

$$\frac{d}{dr}\left[\log\left(r^{1/2}\,J_{t+1/2}\right)\right]_a$$

However, the argument of this Bessel function is $k_1 a$, which is a very large quantity since the conductivity of the earth is so large, and by the asymptotic formula for the Bessel functions of large argument we have

$$\left[\frac{d}{dr}\log\left(r^{1/2}\,J_{t+1/2}\right)\right]_a = k_1\tan\left(k_1 a - t\,\frac{\pi}{2} - \frac{\pi}{2}\right) .$$

Note that this formula is valid only if $k_1 a$ is very much larger than t. Now since k_1 is larger than k, the fraction in the formula (96) must be very small so that

$M_t - M_{-t-1}$ is nearly zero.

If we put t equal to $i\tau - 1/2$ so $-t-1 = -i\tau - 1/2$, we see then that the bracket in the integrand of (39) must be nearly zero for τ very much smaller than $k_1 a$. In the remaining interval of integration we use the formula (40). We find then that that the integrand is of the order of magnitude $e^{-\tau\theta}$. Now

$$\int_{k_a}^{\infty} e^{-\tau\theta}\,d\tau = \frac{1}{\theta}\,e^{-ka\theta}$$

and we see that this is very small compared to $e^{-(ka)^{1/3}\theta}$ or to $e^{-(ka)^{1/4}a}$ which is what is obtained from the residue series. We have thus shown that the integral in (39) is small compared to the first term in the residue series.

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